Similarity Measures between Arguments Revisited

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Abstract. Recently, the notion of similarity between arguments, namely those built using propositional logic, has been investigated and several similarity measures have been defined. This paper shows that those measures may lead to inaccurate results when arguments are not *concise*, i.e., their supports contain information that is useless for inferring their conclusions. For circumventing this limitation, we start by refining arguments for making them concise. Then, we propose two families of similarity measures that extend existing ones and that deal with concise arguments.

Keywords: Logical arguments · Similarity.

1 Introduction

Argumentation is a reasoning process based on the justification of claims by *arguments*. It has received great interest from the Artificial Intelligence community, which used it for solving various problems like decision making (eg., [1, 2]), defeasible reasoning (eg., [3, 4]), handling inconsistency in propositional knowledge bases (eg., [5, 6]), etc.

In case of inconsistency handling, an argument is built from a knowledge base and contains two parts: a *conclusion*, which is a single propositional formula, and a *support*, which is a minimal (for set inclusion) and consistent subset of the base that infers logically the conclusion. Examples of arguments are $A = \langle\{p \wedge q\}, p\rangle$, $B = \langle\{p\}, p\rangle$ and $C = \langle\{p \wedge p\}, p\rangle$. Such arguments may be in conflict and thus an evaluation method, called also semantics in the literature, is used for evaluating their strengths. Some weighting semantics, like h-Categorizer [5], satisfy the Counting (or strict monotony) principle defined in [7]. This principle states that each attacker of an argument contributes to weakening the argument. For instance, if the argument $D = \langle\{\neg p\}, \neg p\rangle$ is attacked by A, B, C, then each of the three arguments will decrease the strength of D. However, the three attackers are somehow similar, thus D will loose more than necessary. Consequently, the authors in [8] have motivated the need for investigating the notion of similarity between pairs of such logical arguments. They introduced a set of principles that a reasonable similarity measure should satisfy, and provided several measures that satisfy them. In [9] the authors introduced three possible extensions of h-Categorizer that take into account similarities between arguments.

While the measures from [8] return reasonable results in most cases, they may lead to inaccurate assessments if arguments are not *concise*. An argument is concise if its support contains only information that is useful for inferring its conclusion. For instance, the argument A is not concise since its support $\{p \wedge q\}$ contains q, which is useless for the conclusion p. Note that minimality of supports does not guarantee conciseness. For example, the support of A is minimal while A is not concise. The similarity measures from [8] declare the two arguments A and B as not fully similar while they support the same conclusion on the same grounds (p). Consequently, both A and B will have an impact on D using h-Categorizer. For circumventing this problem, we propose in this paper to clean up arguments from any useless information. This amounts to generating the concise versions of each argument. The basic idea is to weaken formulas of an argument's support. Then, we apply the measures from [8] on concise arguments in two ways, leading to two different families of measures.

The paper is organized as follows: Section 2 recalls the measures proposed in [8], Section 3 shows how to make arguments concise, Section 4 refines existing measures, and Section 5 concludes and presents some perspectives.

2 Background

We consider classical propositional logic $(\mathcal{L}, \vdash)$, where $\mathcal{L}$ is a propositional language built up from a finite set $\mathcal{P}$ of variables, called atoms, the two Boolean constants $\top$ (true) and $\bot$ (false), and the usual connectives ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$), and $\vdash$ is the consequence relation of the logic. A literal of $\mathcal{L}$ is either a variable of $\mathcal{P}$ or the negation of a variable of $\mathcal{P}$, the set of all literals is denoted by $\mathcal{P}^{\pm}$. A formula ϕ is in negation normal form (NNF) if and only if it does not contain implication or equivalence symbols, and every negation symbol occurs directly in front of an atom. $\mathtt{NNF}(\phi)$ denotes the NNF of ϕ. For instance, $\mathtt{NNF}(\neg((p \rightarrow q) \vee \neg t)) = p \wedge \neg q \wedge t$. $\mathtt{Lit}(\phi)$ denotes the set of literals occurring in $\mathtt{NNF}(\phi)$, hence $\mathtt{Lit}(\neg((p \rightarrow q) \vee \neg t)) = \{p, \neg q, t\}$. Two formulas $\phi, \psi \in \mathcal{L}$ are *logically equivalent*, denoted by $\phi \equiv \psi$, iff $\phi \vdash \psi$ and $\psi \vdash \phi$. In [10], the authors defined the notion of *independence* of a formula from literals as follows.

Definition 1 (Literals Independence). *Let $\phi \in \mathcal{L}$ and $l \in \mathcal{P}^{\pm}$. The formula ϕ is* independent *from the literal l iff $\exists \psi \in \mathcal{L}$ such that $\phi \equiv \psi$ and $l \notin \mathtt{Lit}(\psi)$. Otherwise, ϕ is dependent on l. $\mathtt{DepLit}(\phi)$ denotes the set of all literals that ϕ is dependent on.*

For instance, $\mathtt{DepLit}((\neg p \vee q) \wedge (\neg p \vee \neg q)) = \{\neg p\}$ while $\mathtt{DepLit}(\neg p \wedge q) = \{\neg p, q\}$.

A finite subset Φ of $\mathcal{L}$, denoted by $\Phi \subseteq_f \mathcal{L}$, is *consistent* iff $\Phi \nvdash \bot$, it is *inconsistent* otherwise. Two subsets $\Phi, \Psi \subseteq_f \mathcal{L}$ are *equivalent*, denoted by $\Phi \cong \Psi$, iff $\forall \phi \in \Phi$, $\exists \psi \in \Psi$ such that $\phi \equiv \psi$ and $\forall \psi' \in \Psi$, $\exists \phi' \in \Phi$ such that $\phi' \equiv \psi'$. We write $\Phi \ncong \Psi$ otherwise. This definition is useful in the context of similarity where arguments are compared with respect to their *contents*. Assume, for instance, p and q that stand respectively for "bird" and "fly". Clearly, the two rules "birds fly" and "everything that flies is a bird" express different information. Thus, the two sets $\{p, p \rightarrow q\}$ and $\{q, q \rightarrow p\}$ should be considered as different. Note that $\{p, p \rightarrow q\} \ncong \{q, q \rightarrow p\}$ even if $\mathtt{CN}(\{p, p \rightarrow q\}) = \mathtt{CN}(\{q, q \rightarrow p\})$, where $\mathtt{CN}(\Phi)$ denotes the set of all formulas that follow from the set Φ of formulas.

Let us now recall the backbone of our paper, the notion of logical *argument*.

Definition 2 (Argument). *An* argument *built under the logic* $(\mathcal{L}, \vdash)$ *is a pair* $\langle \Phi, \phi \rangle$, *where* $\Phi \subseteq_f \mathcal{L}$ *and* $\phi \in \mathcal{L}$, *such that:*

- Φ *is consistent,* *(Consistency)*
- $\Phi \vdash \phi$, *(Validity)*
- $\nexists \Phi' \subset \Phi$ *such that* $\Phi' \vdash \phi$. *(Minimality)*

An argument $\langle \Phi, \phi \rangle$ *is* trivial *iff* $\Phi = \emptyset$.

It is worth noticing that trivial arguments support tautologies. It was shown in [11] that the set of arguments that can be built from a finite set of formulas is infinite.

Example 1. The following pairs are all arguments.

$A = \langle \{p \wedge q\}, p \rangle$	$B = \langle \{p\}, p \rangle$
$C = \langle \{p \wedge q \wedge r\}, r \rangle$	$D = \langle \{p \wedge q, p \wedge r\}, p \wedge q \wedge r \rangle$
$E = \langle \{p \wedge q, (p \vee q) \rightarrow r\}, r \rangle$	$F = \langle \{p \wedge q\}, p \vee q \rangle$

Notations: $\texttt{Arg}(\mathcal{L})$ denotes the set of all arguments that can be built under the logic $(\mathcal{L}, \vdash)$. For any $A = \langle \Phi, \phi \rangle \in \texttt{Arg}(\mathcal{L})$, the functions Supp and Conc return respectively the *support* ($\texttt{Supp}(A) = \Phi$) and the *conclusion* ($\texttt{Conc}(A) = \phi$) of A.

In [11], the notion of *equivalence of arguments* has been investigated, and different variants of equivalence have been proposed. The most general one states that two arguments are equivalent if their supports are equivalent (in the sense of $\cong$) and their conclusions are equivalent (in the sense of $\equiv$). For the purpose of our paper, we focus on the following one that requires equality of conclusions.

Definition 3 (Equivalent Arguments). *Two arguments* $A, B \in \texttt{Arg}(\mathcal{L})$ *are* equivalent, *denoted by* $A \approx B$, *iff* $(\texttt{Supp}(A) \cong \texttt{Supp}(B))$ *and* $(\texttt{Conc}(A) = \texttt{Conc}(B))$.

In [8], the authors have investigated the notion of similarity between pairs of arguments, and have introduced several measures which are based on the well-known Jaccard measure [12], Dice measure [13], Sorensen one [14], and those proposed in [15–18]. All these measures compare pairs of non-empty sets (X and Y) of objects. Table 1 shows how to adapt their definitions for comparing supports (respectively conclusions) of arguments, which are sets of propositional formulas. In that table, $\texttt{Co}(\Phi, \Psi)$ is a function that returns for all $\Phi, \Psi \subseteq_f \mathcal{L}$ a set of formulas such that:

$$\texttt{Co}(\Phi, \Psi) = \{\phi \in \Phi \mid \exists \psi \in \Psi \text{ such that } \phi \equiv \psi\}.$$

The definition of each similarity measure between sets of formulas follows the schema below that we illustrate with the Jaccard-based measure. For all $\Phi, \Psi \subseteq_f \mathcal{L}$,

$$s_{\mathtt{j}}(\Phi, \Psi) = \begin{cases} \frac{|\texttt{Co}(\Phi,\Psi)|}{|\Phi| + |\Psi| - |\texttt{Co}(\Phi,\Psi)|} & \text{if } \Phi \neq \emptyset, \Psi \neq \emptyset \\ 1 & \text{if } \Phi = \Psi = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

In [8], a similarity measure between arguments is a function that assigns to every pair of arguments a value from the interval $[0, 1]$. The greater the value, the more similar are the arguments. Such measure should satisfy some properties including symmetry.

Extended Jaccard	$s_{\mathtt{j}}(\Phi,\Psi) = \dfrac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert + \lvert\Psi\rvert - \lvert\mathtt{Co}(\Phi,\Psi)\rvert}$
Extended Dice	$s_{\mathtt{d}}(\Phi,\Psi) = \dfrac{2\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert + \lvert\Psi\rvert}$
Extended Sorensen	$s_{\mathtt{s}}(\Phi,\Psi) = \dfrac{4\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert + \lvert\Psi\rvert + 2\lvert\mathtt{Co}(\Phi,\Psi)\rvert}$
Extended Symmetric Anderberg	$s_{\mathtt{a}}(\Phi,\Psi) = \dfrac{8\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert + \lvert\Psi\rvert + 6\lvert\mathtt{Co}(\Phi,\Psi)\rvert}$
Extended Sokal and Sneath 2	$s_{\mathtt{ss}}(\Phi,\Psi) = \dfrac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{2(\lvert\Phi\rvert + \lvert\Psi\rvert) - 3\lvert\mathtt{Co}(\Phi,\Psi)\rvert}$
Extended Ochiai	$s_{\mathtt{o}}(\Phi,\Psi) = \dfrac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\sqrt{\lvert\Phi\rvert}\sqrt{\lvert\Psi\rvert}}$
Extended Kulczynski 2	$s_{\mathtt{ku}}(\Phi,\Psi) = \dfrac{1}{2}\left(\dfrac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert} + \dfrac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Psi\rvert}\right)$

Table 1. Similarity Measures for Sets $\Phi, \Psi \subseteq_f \mathcal{L}$.

Definition 4 (Similarity Measure). *A* similarity measure *is a function* $\mathcal{S} : \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L}) \to [0,1]$ *such that:*

Symmetry: for all $a, b \in \mathtt{Arg}(\mathcal{L})$*,* $\mathcal{S}(a,b) = \mathcal{S}(b,a)$*.*
Maximality: for any $a \in \mathtt{Arg}(\mathcal{L})$*,* $\mathcal{S}(a,a) = 1$*.*
Substitution: for all $a, b, c \in \mathtt{Arg}(\mathcal{L})$*, if* $\mathcal{S}(a,b) = 1$ *then* $\mathcal{S}(a,c) = \mathcal{S}(b,c)$*.*

In [8], several similary measures have been defined. They apply any measure from Table 1 for assessing similarity of both arguments' supports and their conclusions. Furthermore, they use a parameter that allows a user to give different importance degrees to the two components of an argument. Those measures satisfy additional properties (see [8] for more details).

Definition 5 (Extended Measures). *Let* $0 < \sigma < 1$ *and* $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$*. A similarity measure* $\mathcal{S}_x^\sigma$ *is a function assigning to any pair* $(A,B) \in \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$ *a value* $\mathcal{S}_x^\sigma(A,B) = \sigma \cdot s_x(\mathtt{Supp}(A), \mathtt{Supp}(B)) + (1-\sigma) \cdot s_x(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\})$*.*

Example 1 (Continued). Let $\sigma = 0.5$ and $x = \mathtt{j}$.

- $\mathcal{S}_{\mathtt{j}}^{0.5}(A,B) = 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$
- $\mathcal{S}_{\mathtt{j}}^{0.5}(A,D) = 0.5 \cdot 0.5 + 0.5 \cdot 0 = 0.25$
- $\mathcal{S}_{\mathtt{j}}^{0.5}(A,F) = 0.5 \cdot 1 + 0.5 \cdot 0 = 0.5$

3 Concise Arguments

The two arguments $A = \langle \{p \wedge q\}, p \rangle$ and $B = \langle \{p\}, p \rangle$ are not fully similar according to the existing measures while they support the same conclusion and on the same grounds. This is due to the non-conciseness of A, which contains the useless information q in its support. In what follows, we refine arguments by removing from their supports such information. The idea is to weaken formulas in supports.

Definition 6 (Refinement). *Let $A, B \in \texttt{Arg}(\mathcal{L})$ such that $A = \langle\{\phi_1, \ldots, \phi_n\}, \phi\rangle$ and $B = \langle\{\phi'_1, \ldots, \phi'_n\}, \phi'\rangle$. B is a* refinement *of A iff:*

1. $\phi = \phi'$,
2. *There exists a permutation ρ of the set $\{1, \ldots, n\}$ such that $\forall k \in \{1, \ldots, n\}$, $\phi_k \vdash \phi'_{\rho(k)}$ and $\texttt{Lit}(\phi'_{\rho(k)}) \subseteq \texttt{DepLit}(\phi_k)$.*

Let Ref *be a function that returns the set of all refinements of a given argument.*

The second condition states that each formula of an argument's support is weakened. Furthermore, novel literals are not allowed in the weakening step since such literals would negatively impact similarity between supports of arguments. Finally, literals from which a formula is independent should be removed since they are useless for inferring the conclusion of an argument. It is worth mentioning that an argument may have several refinements as shown in the following example.

Example 1 (Continued).

- $\{\langle\{p\}, p\rangle, \langle\{p \wedge p\}, p\rangle\} \subseteq \texttt{Ref}(A)$
- $\{\langle\{p \wedge r\}, r\rangle, \langle\{q \wedge r\}, r\rangle, \langle\{r\}, r\rangle\} \subseteq \texttt{Ref}(C)$
- $\{\langle\{p \wedge q, r\}, p \wedge q \wedge r\rangle, \langle\{q, p \wedge r\}, p \wedge q \wedge r\rangle\} \subseteq \texttt{Ref}(D)$
- $\{\langle\{p \vee q, (p \vee q) \rightarrow r\}, r\rangle, \langle\{p, p \rightarrow r\}, r\rangle, \langle\{q, q \rightarrow r\}, r\rangle\} \subseteq \texttt{Ref}(E)$
- $\{\langle\{p\}, p \vee q\rangle, \langle\{q\}, p \vee q\rangle, \langle\{p \vee q\}, p \vee q\rangle\} \subseteq \texttt{Ref}(F)$

The following property shows that there exists a unique possible permutation ρ for each refinement of an argument.

Proposition 1. *For all $A = \langle\{\phi_1, \ldots, \phi_n\}, \phi\rangle, B = \langle\{\phi'_1, \ldots, \phi'_n\}, \phi'\rangle \in \texttt{Arg}(\mathcal{L})$ such that $B \in \texttt{Ref}(A)$, there exists a unique permutation ρ of the set $\{1, \cdots, n\}$ such that $\forall k \in \{1, \ldots, n\}$, $\phi_k \vdash \phi'_{\rho(k)}$.*

A trivial argument is the only refinement of itself.

Proposition 2. *For any trivial argument $A \in \texttt{Arg}(\mathcal{L})$, $\texttt{Ref}(A) = \{A\}$.*

A non-trivial argument has a non-empty set of refinements. Moreover, it is a refinement of itself only if the formulas of its support do not contain literals from which they are independent.

Proposition 3. *Let $A \in \texttt{Arg}(\mathcal{L})$ be a non-trivial argument. The following hold:*

- $\texttt{Ref}(A) \neq \emptyset$,
- $A \in \texttt{Ref}(A)$ *iff* $\forall \phi \in \texttt{Supp}(A)$, $\texttt{Lit}(\phi) = \texttt{DepLit}(\phi)$.

We show next that the function Ref is idempotent and that equivalent arguments have the same refinements.

Proposition 4. *Let $A, B \in \texttt{Arg}(\mathcal{L})$. The following hold:*

- *If $B \in \texttt{Ref}(A)$, then $\texttt{Ref}(B) \subseteq \texttt{Ref}(A)$.*

– *If* $A \approx B$*, then* $\texttt{Ref}(A) = \texttt{Ref}(B)$.

We are now ready to define the backbone of the paper, the novel notion of concise argument. An argument is concise if it is equivalent to any of its refinements. This means that a concise argument cannot be further refined.

Definition 7 (Conciseness). *An argument* $A \in \texttt{Arg}(\mathcal{L})$ *is* concise *iff for all* $B \in \texttt{Ref}(A)$*,* $A \approx B$.

Example 1 (Continued). The two refinements $\langle\{p \wedge r\}, r\rangle$ and $\langle\{q \wedge r\}, r\rangle$ of the argument C are not concise. Indeed, $\langle\{r\}, r\rangle \in \texttt{Ref}(\langle\{p \wedge r\}, r\rangle)$, $\langle\{r\}, r\rangle \in \texttt{Ref}(\langle\{q \wedge r\}, r\rangle)$ while $\langle\{r\}, r\rangle \not\approx \langle\{p \wedge r\}, r\rangle$, and $\langle\{r\}, r\rangle \not\approx \langle\{q \wedge r\}, r\rangle$.

For any argument from $\texttt{Arg}(\mathcal{L})$, we generate its concise versions. The latter are simply its concise refinements.

Definition 8 (Concise Refinements). *A concise refinement of an argument* $A \in \texttt{Arg}(\mathcal{L})$ *is any concise argument* B *such that* $B \in \texttt{Ref}(A)$*. We denote the set of all concise refinements of* A *by* $\texttt{CR}(A)$.

Example 1 (Continued).

– $\langle\{p\}, p\rangle \in \texttt{CR}(A)$
– $\langle\{r\}, r\rangle \in \texttt{CR}(C)$
– $\{\langle\{p \wedge q, r\}, p \wedge q \wedge r\rangle, \langle\{q, p \wedge r\}, p \wedge q \wedge r\rangle\} \subseteq \texttt{CR}(D)$
– $\{\langle\{p \vee q, (p \vee q) \rightarrow r\}, r\rangle, \langle\{p, p \rightarrow r\}, r\rangle, \langle\{q, q \rightarrow r\}, r\rangle\} \subseteq \texttt{CR}(E)$
– $\{\langle\{p\}, p \vee q\rangle, \langle\{q\}, p \vee q\rangle, \langle\{p \vee q\}, p \vee q\rangle\} \subseteq \texttt{CR}(F)$

Next we state some properties of concise refinements.

Proposition 5. *Let* $A \in \texttt{Arg}(\mathcal{L})$*. The following hold:*

1. *For any* $B \in \texttt{CR}(A)$ *the following hold:* $B \in \texttt{Ref}(B)$ *and* $\forall C \in \texttt{Ref}(B)$*,* $C \approx B$.
2. $\texttt{CR}(A) \neq \emptyset$.
3. *If* A *is non-trivial, then* $\texttt{CR}(A)$ *is infinite.*
4. *If* $A \approx B$*, then* $\texttt{CR}(A) = \texttt{CR}(B)$.
5. $\forall B \in \texttt{Ref}(A)$*,* $\texttt{CR}(B) \subseteq \texttt{CR}(A)$.

The following result shows that any formula in the support of a concise argument cannot be further weakened without introducing additional literals.

Proposition 6. *Let* $A, B \in \texttt{Arg}(\mathcal{L})$ *such that* $B \in \texttt{CR}(A)$*. For any* $\phi \in \texttt{Supp}(B)$*, if* $\exists \psi \in \mathcal{L}$ *such that* $\phi \vdash \psi$*,* $\psi \not\vdash \phi$*, and* $\langle(\texttt{Supp}(B) \setminus \{\phi\}) \cup \{\psi\}, \texttt{Conc}(B)\rangle \in \texttt{Arg}(\mathcal{L})$*, then* $\texttt{Lit}(\psi) \setminus \texttt{Lit}(\phi) \neq \emptyset$.

4 Similarity Measures

As already said in previous sections, although the similarity measures from Definition 5 return reasonable results in most cases, they might lead to inaccurate assessments if the arguments are not concise. Indeed, as we illustrated in Section 2, the measures from Definition 5 declare the two arguments $A = \langle\{p \wedge q\}, p\rangle$ and $B = \langle\{p\}, p\rangle$ as not fully similar, while they support the same conclusion based on the same grounds (p).

In this section, we extend those measures in two ways, leading to two families of similarity measures, using concise refinements of arguments, and we show that they properly resolve the drawbacks of the existing measures. Note that by Proposition 5(3), every non-trivial argument A has infinitely many concise refinements. This is due to the fact that every formula α from a support of a concise refinement can be equivalently rewritten in infinitely many ways using the same set of literals (eg. $\alpha \equiv \alpha \wedge \alpha \equiv \alpha \wedge \alpha \wedge \alpha \equiv \cdots$). In the rest of the paper, we will consider only one argument from $\mathtt{CR}(A)$ per equivalence class. For that reason, we consider a fixed set $\overline{\mathcal{L}} \subset \mathcal{L}$ such that $\phi \in \mathcal{L}$ there exists a unique $\psi \in \overline{\mathcal{L}}$ such that $\psi \equiv \phi$. Furthermore, we assume that each $\psi \in \overline{\mathcal{L}}$ contains only dependent literals.

Definition 9. *Let $A \in \mathtt{Arg}(\mathcal{L})$. We define the set*

$$\overline{\mathtt{CR}}(A) = \{B \in \mathtt{CR}(A) \mid \mathtt{Supp}(B) \subset \overline{\mathcal{L}}\}.$$

In this way, we obtain a finite set of non-equivalent concise refinements.

Proposition 7. *For every $A \in \mathtt{Arg}(\mathcal{L})$, the set $\overline{\mathtt{CR}}(A)$ is finite.*

Now we propose our first family of similarity measures. In the following definition, for $A \in \mathtt{Arg}(\mathcal{L})$, $\Sigma \subseteq_f \mathtt{Arg}(\mathcal{L})$ and a similarity measure S from Definition 5, we denote by $\mathtt{Max}(A, \Sigma, S)$ the maximal similarity value between A and an argument from Σ according to S, i.e.,

$$\mathtt{Max}(A, \Sigma, S) = \max_{B \in \Sigma} S(A, B).$$

The first family of measures compares the sets of concise refinements of the two arguments under study. Indeed, the similarity between A and B is the average of maximal similarities (using any existing measure from Definition 5) between any concise refinement of A and those of B.

Definition 10 (A-CR Similarity Measures). *Let $A, B \in \mathtt{Arg}(\mathcal{L})$, and let S be a similarity measure from Definition 5. We define* A-CR *similarity measure*[3] *by*

$$s^{\mathtt{A}}_{\mathtt{CR}}(A, B, S) = \frac{\sum\limits_{A_i \in \overline{\mathtt{CR}}(A)} \mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), S) + \sum\limits_{B_j \in \overline{\mathtt{CR}}(B)} \mathtt{Max}(B_j, \overline{\mathtt{CR}}(A), S)}{|\overline{\mathtt{CR}}(A)| + |\overline{\mathtt{CR}}(B)|}.$$

The value of A-CR similarity measure always belongs to the unit interval.

Proposition 8. *Let $A, B \in \mathtt{Arg}(\mathcal{L})$, S^{σ}_x a similarity measure where $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $0 < \sigma < 1$. Then $s^{\mathtt{A}}_{\mathtt{CR}}(A, B, S^{\sigma}_x) \in [0, 1]$.*

[3] A in A-CR stands for "average".

Next we show that the new measure properly resolves the problem of non-conciseness of the argument $A = \langle \{p \wedge q\}, p \rangle$ from our running example. We illustrate that by considering Extended Jaccard measure with the parameter $\sigma = 0.5$.[4]

Example 1 (Continued). It is easy to check that $\overline{\mathtt{CR}}(A) = \{\langle \{p\}, p \rangle\}$ and $\overline{\mathtt{CR}}(B) = \{\langle \{p\}, p \rangle\}$. Then $s^{\mathtt{A}}_{\mathtt{CR}}(A, B, S^{0.5}_{\mathtt{j}}) = 1$ while $\mathtt{S}^{0.5}_{\mathtt{j}}(A, B) = 0.5$.

Now we define our second family of similarity measures, which is based on comparison of sets obtained by merging supports of concise refinements of arguments. For an argument $A \in \mathtt{Arg}(\mathcal{L})$, we denote that set by

$$\mathtt{US}(A) = \bigcup_{A' \in \overline{\mathtt{CR}}(A)} \mathtt{Supp}(A').$$

Definition 11 **($\mathtt{U\text{-}CR}$ Similarity Measures).** *Let $A, B \in \mathtt{Arg}(\mathcal{L})$, $0 < \sigma < 1$, and s_x be a similarity measure from Table 1. We define* $\mathtt{U\text{-}CR}$ *similarity measure*[5] *by*

$$s^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_x, \sigma) = \sigma \cdot s_x(\mathtt{US}(A), \mathtt{US}(B)) + (1 - \sigma) \cdot s_x(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\}).$$

Next example illustrates that $\mathtt{U\text{-}CR}$ also properly resolves the problem of non-conciseness of the argument $A = \langle \{p \wedge q\}, p \rangle$ from our running example.

Example 1 (Continued). Let $\sigma = 0.5$ and $x = \mathtt{j}$. It is easy to check that $s^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{\mathtt{j}}, 0.5) = 1$ while $\mathtt{S}^{0.5}_{\mathtt{j}}(A, B) = 0.5$.

Let us now consider another more complex example where existing similarity measures provide inaccurate values while the new ones perform well.

Example 2. Let us consider the following arguments:

- $A = \langle \{p \wedge q, (p \vee q) \to t, (p \vee t) \to r\}, t \wedge r \rangle$
- $B = \langle \{p, p \to t, p \to r\}, t \wedge r \rangle$

It is easy to check that $\overline{\mathtt{CR}}(A) = \{A_1, A_2, A_3, A_4, A_5\}$ and $\overline{\mathtt{CR}}(B) = \{B_1\}$, where:

- $A_1 = \langle \{p, p \to t, p \to r\},\ t \wedge r \rangle$
- $A_2 = \langle \{p, p \to t, t \to r\},\ t \wedge r \rangle$
- $A_3 = \langle \{q, q \to t, t \to r\}, t \wedge r \rangle$
- $A_4 = \langle \{p \vee q, (p \vee q) \to t, t \to r\},\ t \wedge r \rangle$
- $A_5 = \langle \{p \wedge q, q \to t, p \to r\},\ t \wedge r \rangle$
- $B_1 = \langle \{p, p \to t, p \to r\}, t \wedge r \rangle$

It is worth noticing that the Extended Jaccard measure could not detect any similarity between the supports of A and B while their concise arguments A_1 and B_1 are identical. Indeed, $s_{\mathtt{j}}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = 0$ and $S^{0.5}_{\mathtt{j}}(A, B) = 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$ while $s^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{\mathtt{j}}, 0.5) = 0.5 \cdot \frac{3}{9} + 0.5 \cdot 1 = \frac{2}{3} = 0.666$ and $s^{\mathtt{A}}_{\mathtt{CR}}(A, B, S^{0.5}_{\mathtt{j}}) = 0.5 \cdot \frac{9}{20} + 0.5 \cdot 1 = \frac{29}{40} = 0.725$.

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[4] In this section, we slightly relax the notation by simply assuming that $p \in \overline{\mathcal{L}}$. We will make similar assumptions throughout this section.

[5] $\mathtt{U}$ in $\mathtt{U\text{-}CR}$ stands for "union".

The following proposition characterizes the arguments which are fully similar according to the novel measures. It states that full similarity is obtained exactly in the case when two arguments have equivalent concise refinements.

Proposition 9. *Let* $A, B \in \texttt{Arg}(\mathcal{L})$*,* $0 < \sigma < 1$ *and* $x \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}\}$*. Then* $s^{\texttt{A}}_{\texttt{CR}}(A, B, S^{\sigma}_{x}) = s^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, \sigma) = 1$ *iff:*

– $\forall A' \in \overline{\texttt{CR}}(A)$*,* $\exists B' \in \overline{\texttt{CR}}(B)$ *such that* $\texttt{Supp}(A') \cong \texttt{Supp}(B')$*,* $\texttt{Conc}(A') \equiv \texttt{Conc}(B')$ *and*
– $\forall B' \in \overline{\texttt{CR}}(B)$*,* $\exists A' \in \overline{\texttt{CR}}(A)$ *such that* $\texttt{Supp}(B') \cong \texttt{Supp}(A')$*,* $\texttt{Conc}(B') \equiv \texttt{Conc}(A')$*.*

In [8], the authors proposed a set of principles that a reasonable similarity measure should satisfy. Now we show that the new measures satisfy four of them but violate Monotony. The reason of violation is due to the definition itself of the principle. Indeed, it is based on the supports of arguments. The new measures do not handle those supports but those of the concise refinements of the initial arguments.

Proposition 10. *Let* $0 < \sigma < 1$ *and* $x \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}\}$*. The following hold:*

(Syntax Independence) *Let* π *be a permutation on the set of variables, and* A, B, A', $B' \in \texttt{Arg}(\mathcal{L})$ *such that*
- A' *is obtain by replacing each variable* p *in* A *with* $\pi(p)$*,*
- B' *is obtain by replacing each variable* p *in* B *with* $\pi(p)$*.*

Then $s^{\texttt{A}}_{\texttt{CR}}(A, B, S^{\sigma}_{x}) = s^{\texttt{A}}_{\texttt{CR}}(A', B', S^{\sigma}_{x})$ *and* $s^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, \sigma) = s^{\texttt{U}}_{\texttt{CR}}(A', B', s_x, \sigma)$*.*
(Maximality) *For every* $A \in \texttt{Arg}(\mathcal{L})$*,* $s^{\texttt{A}}_{\texttt{CR}}(A, A, S^{\sigma}_{x}) = s^{\texttt{U}}_{\texttt{CR}}(A, A, s_x, \sigma) = 1$*.*
(Symmetry) *For all* $A, B \in \texttt{Arg}(\mathcal{L})$*,* $s^{\texttt{A}}_{\texttt{CR}}(A, B, S^{\sigma}_{x}) = s^{\texttt{A}}_{\texttt{CR}}(B, A, S^{\sigma}_{x})$ *and* $s^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, \sigma) = s^{\texttt{U}}_{\texttt{CR}}(B, A, s_x, \sigma)$*.*
(Substitution) *For all* $A, B, C \in \texttt{Arg}(\mathcal{L})$*,*
- *if* $s^{\texttt{A}}_{\texttt{CR}}(A, B, S^{\sigma}_{x}) = 1$*, then* $s^{\texttt{A}}_{\texttt{CR}}(A, C, S^{\sigma}_{x}) = s^{\texttt{A}}_{\texttt{CR}}(B, C, S^{\sigma}_{x})$*,*
- *if* $s^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, \sigma) = 1$*, then* $s^{\texttt{U}}_{\texttt{CR}}(A, C, s_x, \sigma) = s^{\texttt{U}}_{\texttt{CR}}(B, C, s_x, \sigma)$*.*

The next proposition shows that if we apply A-CR or U-CR to any similarity measure S^{σ}_{x} from Definition 5 (respectively s_x from Table 1), then both novel measures will coincide with S^{σ}_{x} on the class of concise arguments.

Proposition 11. *Let* $A, B \in \texttt{Arg}(\mathcal{L})$ *be two concise arguments. Then, for every* $0 < \sigma < 1$ *and* $x \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}\}$*, it holds*

$$s^{\texttt{A}}_{\texttt{CR}}(A, B, S^{\sigma}_{x}) = s^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, \sigma) = S^{\sigma}_{x}(A, B). \tag{1}$$

Remark. Note that the equations (1) might also hold for some A and B that are not concise. For example, let $A = \langle \{p \wedge q, t \wedge s\}, p \wedge t \rangle$ and $B = \langle \{p, t \wedge s\}, p \wedge s \rangle$. Then $\overline{\texttt{CR}}(A) = \{\langle \{p, t\}, p \wedge t \rangle\}$ and $\overline{\texttt{CR}}(B) = \{\langle \{p, s\}, p \wedge s \rangle\}$, so $s^{\texttt{A}}_{\texttt{CR}}(A, B, S^{0.5}_{\texttt{j}}) = s^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{j}}, 0.5) = S^{0.5}_{\texttt{j}}(A, B) = 0.25$.

The following example shows that A-CR and U-CR may return different results. Indeed, it is possible for three arguments A, B and C that A is more similar to B than to C according to one measure, but not according to the other one.

Example 3. Let $A = \langle\{p, p \rightarrow q_1 \wedge q_2\}, q_1 \vee q_2\rangle$, $B = \langle\{p, s\}, p \wedge s\rangle$ and $C = \langle\{p \rightarrow q_1\}, p \rightarrow q_1\rangle$. We have $\overline{\mathtt{CR}}(A) = \{\langle\{p, p \rightarrow q_1\}, q_1 \vee q_2\rangle, \langle\{p, p \rightarrow q_2\}, q_1 \vee q_2\rangle, \langle\{p, p \rightarrow q_1 \vee q_2\}, q_1 \vee q_2\rangle\}$, $\overline{\mathtt{CR}}(B) = \{\langle\{p, s\}, p \wedge s\rangle\}$, $\overline{\mathtt{CR}}(C) = \{\langle\{p \rightarrow q_1\}, p \rightarrow q_1\rangle\}$. Consequently:

– $s^{\mathtt{A}}_{\mathtt{CR}}(A, B, S^{0.5}_{\mathtt{j}}) = \frac{1}{6} > s^{\mathtt{A}}_{\mathtt{CR}}(A, C, S^{0.5}_{\mathtt{j}}) = \frac{1}{8}$, but
– $s^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{\mathtt{j}}, 0.5) = \frac{1}{10} < s^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_{\mathtt{j}}, 0.5) = \frac{1}{8}$.

The next example shows that none of the two novel measures dominates the other. Indeed, some pairs of arguments have greater similarity value according to A-CR, and other pairs have greater similarity value using U-CR.

Example 3 (Continued). Note that $s^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{\mathtt{j}}, 0.5) < s^{\mathtt{A}}_{\mathtt{CR}}(A, B, S^{0.5}_{\mathtt{j}})$. Let us consider $A' = \langle\{p \wedge q\}, p \vee q\rangle$, $B' = \langle\{p, q\}, p \wedge q\rangle \in \mathtt{Arg}(\mathcal{L})$. $s^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_{\mathtt{j}}, 0.5) = 0.5 \cdot \frac{2}{3} + 0.5 \cdot 0 = \frac{1}{3} = 0.333$ and $s^{\mathtt{A}}_{\mathtt{CR}}(A', B', S^{0.5}_{\mathtt{j}}) = 0.5 \cdot \frac{3}{8} + 0.5 \cdot 0 = \frac{3}{16} = 0.1875$, thus $s^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_{\mathtt{j}}, 0.5) > s^{\mathtt{A}}_{\mathtt{CR}}(A', B', S^{0.5}_{\mathtt{j}})$.

5 Conclusion

The paper tackled the question of similarity between logical arguments. Starting from the observation that existing similarity measures may provide inaccurate assessments, the paper investigated the origin of this limitation and showed that it is due to the presence of useless information in the supports of arguments. It then introduced the novel notion of concise argument, and a procedure for generating the concise versions of any argument. These versions are then used together with existing similarity measures for extending the latter into more efficient measures.

This work can be extended in different ways. The first one consists of identifying a principle, or formal property for distinguishing the new measures. The second one consists of investigating other approaches for generating concise arguments, namely we plan to use the well-known forgetting operator for getting rid of useless literals in formulas. The Third one consists of using the new measures for refining argumentation systems that deal with inconsistent information. Finally, we plan to investigate the notion of similarity for other types of arguments, like analogical arguments.

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